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Calculating dynamic structure factors with the real space renormalisation group

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Abstract. This paper develops exact methods, based on the real space renormalisation group, for calculating dynamic structure factors of Heisenberg ferromagnets. The method is applied to a pure chain, a Sierpinski gasket and a quasi-one-dimensional model of a dilute chain. The result for the Sierpinski gasket exhibits a rich structure because of the requirement of discrete scale invariance.

By the use of distribution scaling the method is generalised to treat a randomly diluted chain.

1. Introduction

In recent years much interest has been shown in the anomalous dynamics of systems close to dilution-induced critical points, such as those occurring in dilute magnets close to the percolation threshold. In this paper we shall concentrate on the linear dynamics of systems with a Goldstone symmetry, such as the dilute Heisenberg ferromagnet. Analogous anomalous effects occur in other systems and have also been widely considered (Alexander and Orbach 1982, Rammal and Toulouse 1982). Recent reviews can be found by Aharony (1985) and Orbach (1985).

It is expected that, close to a dilution-induced critical point, the Green functions (and hence the correlation functions) take on the following homogeneous form (Halperin and Hohenberg 1969):

$$G(k, \omega, \xi) = \langle\langle S_k^+; S_{-k}^- \rangle\rangle_\omega \tag{1.1}$$

$$= k^{(2-\eta+z)} F(\omega/k^z, k\xi) \tag{1.2}$$

where ξ is the static percolative correlation length. From $G(k, \omega)$ we find the dynamic structure factor, $S(k, \omega, \xi)$, which can be observed in scattering experiments:

$$S(k, \omega, \xi) = \lim_{\varepsilon \to 0} \operatorname{Im} G(k, \omega - i\varepsilon, \xi). \tag{1.3}$$

The form (1.2) has been confirmed in one dimension by Stinchcombe and Harris (1983) and Maggs and Stinchcombe (1984) who calculated the function $F(\omega/k^z, k\xi)$ in closed form for a dilute ferromagnetic and antiferromagnetic chain respectively. In more than one dimension the methods used in those papers cannot be used. For small amounts of dilution it is possible to use classical effective medium theories (Tahir-Kheli

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1972, Elliott and Pepper 1973, Entin-Wohlman *et al* 1983, Derrida *et al* 1984). Lewis and Stinchcombe (1984) considered the density of states of a dilute Heisenberg ferromagnet at the percolation threshold, where $\xi = \infty$, by a numerical study of the interaction matrices of large randomly generated clusters.

To deal with the scaling aspects of critical dynamics at the percolation threshold, in one and more dimensions, workers have been led to consider the application of real space methods to the dynamics of random media. For general reviews see Stinchcombe (1983b, c, 1985). These methods have been applied to random systems by a number of workers. Stinchcombe (1983a) and Stinchcombe and Harris (1983) consider a randomly diluted chain. The methods were applied to a two-dimensional hexagonal lattice at the percolation threshold by Harris and Stinchcombe (1983). Here, the standard bond moving and moment averaging methods used in the theory of dilute magnetism (Stinchcombe 1983b) were adapted to treat random equations of motion and find the scaling form and the exponent z.

Real space methods have been applied to a random binary alloy by Gonçalves da Silva and Koiller (1981) who produced a procedure for calculating the density of states by a partial averaging over the disorder in the equations of motion. The results have since been re-derived and extended by Robbins and Koiller (1983) so that a partial average is performed over the t matrices; this was found to increase the number of moments of the density of states that are preserved from five, in the original approximation, to seven. The method is also capable of dealing with Markovian short range order. Similar methods have also been used for a disordered ferromagnet (Chaves and Koiller 1983). The success of the original approximation in ensuring a positive definite density of states has been shown to lie in the fact that the method can be regarded as an exact recursive solution for the density of states of a Berker lattice with an inhomogeneous distribution of masses (Langlois *et al* 1983).

Real space methods are ideally suited to finding the exact solution of self-similar or fractal models. Fractals (Mandelbrot 1977) have been of interest to physicists since they can be constructed to model relevant features of random systems. For dilute lattices at the percolation threshold the important characteristics are thought to be statistical self-similarity and a highly ramified structure.

One of the most popular models has been the Sierpinski gasket upon which both thermodynamics and dynamics have been considered. The papers of Harris and Stinchcombe (1983) and Domany *et al* (1983) considered the dynamics of a Heisenberg spin system on the Sierpinski gasket and used methods based on scaling the equations of motion of the spins. Domany *et al* (1983) showed that it is possible to find the density of states by considering the quadratic map

$$\omega' = 5\omega - \omega^2. \tag{1.4}$$

They also considered the nature of eigenstates on the gasket and concluded that, rather than being extended, as is usual on Euclidean lattices, they are rather local molecular modes which do not necessarily extend throughout the whole lattice. Using the equation of motion method Harris and Stinchcombe (1983) also considered the scaling of Green functions on the gasket.

Tremblay and Southern (1983) produced another method of calculating the density of states of the Sierpinski gasket by introducing a Gaussian generating function for the density of states. From this generating function they derived an inhomogeneous functional equation. The method has much in common with real space calculations of the specific heats of magnets (Niemeijer and van Leeuwen 1976). By iterating the

functional equation in the complex plane, and by giving ω a small imaginary part, the full density of states was found. The advantage of this method over the method of Domany *et al* (1983) is that it is no longer necessary to treat some frequencies as special cases: the singularities which arise in the application of (1.4) are smoothed out by the finite imaginary part of ω.

Southern and Douchant (1985) have also applied this method to a series of identical Sierpinski gaskets embedded in a triangular lattice. This system is proposed to model a dilute two-dimensional lattice close to p_c. By changing the generation of the gaskets which are inserted into the triangular lattice it is possible to change the effective correlation length. If we believe the dynamic scaling hypothesis, the side of the gasket, measured in units of the smallest triangle present, should be the only relevant length for low frequency dynamics. Surprisingly they found that, rather than there being an extended band at low energies with a crossover to localised states at higher energies, when anomalous dispersion sets in, there is instead an alternation of narrow extended bands and localised modes throughout the whole energy range of excitations. A finite fraction of the modes always remains extended no matter how large the correlation length becomes. Whether these high energy extended states are an artefact of the periodic array of gaskets or are instead also present in random systems is not known. It may be that in a random cluster the energies of these high energy pass bands vary sufficiently for there to be a mismatch between neighbouring regions, so that only the lowest band (which is constrained by the Goldstone symmetry) remains extended.

This paper considers the generalisation of these methods so that they can be used to calculate the full response function of Heisenberg ferromagnets at zero temperature. The methods developed are applied to a pure chain, where the exact solution is recovered, a Sierpinski gasket and a quasi-one-dimensional model consisting of a linear backbone decorated with side chains. This last system has been designed to model some of the properties of a dilute ferromagnetic chain near the percolation threshold, and yet avoid the difficulties associated with making arbitrary truncations to recursion formulae.

It would also be useful to have a method of calculating $G(k, \omega)$ for random systems with real space methods which could be applied for p close to p_c. This would complement the results of effective medium calculations which are only reliable near the pure limit. For any random system one would expect to make approximations to the true renormalised coupling distributions, both to limit the range of interactions and to remove correlations which develop in the couplings in the process of renormalisation. Unfortunately we have been unable to find any completely acceptable approximations to the general problem of Heisenberg dynamic response on a dilute d-dimensional lattice. The major difficulty encountered in making any approximation is discussed later in § 7 when we consider the convergence of an exact recursion formula for the Green functions of a ferromagnetic chain.

Finally we calculate the response of a chain segment of 2^p bonds which leads us to an exact treatment of the randomly diluted chain by the use of distribution scaling.

2. The Gaussian generating function

Consider a Heisenberg spin system on an arbitrary lattice. Then the equations of motion of the Green functions

$$G_{ij} = \langle\langle S_i^+; S_j^- \rangle\rangle \tag{2.1}$$

can be found by the usual method of commutation with the Hamiltonian. They are as follows:

$$\left(\sum_l J_{il} - \omega\right) G_{ij} = \sum_l J_{il} G_{lj} + \delta_{ij}. \tag{2.2}$$

It is convenient to write these equations in matrix form as

$$\mathbf{HG} = \mathbf{I} \tag{2.3}$$

where

$$\mathbf{H}_{ii} = \sum_l J_{il} - \omega \qquad \mathbf{H}_{ij} = J_{ij}. \tag{2.4}$$

We now see that the Green functions G_{ij} are given by the inverse of the matrix, $\mathbf{H}$. We shall now use a recursive method of calculating this inverse by introducing a Gaussian generating function.

We define the 'free energy' $F\{h\}$ by

$$F\{h\} = \ln \int \exp[(\mathrm{i}/2)(\boldsymbol{S}\mathbf{H}\boldsymbol{S} + 2\boldsymbol{h}\cdot\boldsymbol{S})] \prod_l dS_l. \tag{2.5}$$

The action in the exponential is an abbreviation for

$$\tfrac{1}{2}\sum_{\langle ij\rangle} J_{ij}(S_i - S_j)^2 - \omega \sum_i S_i^2 + 2\sum_i h_i S_i. \tag{2.6}$$

As is easily seen from completing the square in the exponential

$$F\{h\} = -\tfrac{1}{2}\mathrm{i}\sum_{ij} h_i G_{ij} h_j + \text{constant} \tag{2.7}$$

where the matrix $\mathbf{G}$ is again the inverse of the matrix $\mathbf{H}$ defined by (2.4), which is the matrix of the symmetric quadratic form occurring in the action (2.6). The individual elements of $\mathbf{G}$ are found by differentiating with respect to the transverse magnetic fields, h:

$$G_{ij} = \mathrm{i}\frac{\partial^2}{\partial h_i \partial h_j} F\{h\}. \tag{2.8}$$

The advantage of introducing a Gaussian generating function is that the analogy between the dynamics of Heisenberg magnets and thermodynamics becomes much clearer. We shall now use (2.8) to derive some formal results for the renormalisation of parameters occurring in an arbitrary action of the form (2.6). To do this we split the action into two parts corresponding to two sublattices. The sites which are to remain after a single renormalisation has been performed are assigned to the sublattice labelled '1'. The remaining sites, which are to be integrated over, form the '2' sublattice. The action can be rewritten in terms of the sublattice variables and the matrices that connect them as follows:

$$\boldsymbol{S}\mathbf{H}\boldsymbol{S} + 2\boldsymbol{h}\boldsymbol{S} = \boldsymbol{S}_1\mathbf{H}_{11}\boldsymbol{S}_1 + \boldsymbol{S}_1\mathbf{H}_{12}\boldsymbol{S}_2 + \boldsymbol{S}_2\mathbf{H}_{21}\boldsymbol{S}_2 + \boldsymbol{S}_2\mathbf{H}_{22}\boldsymbol{S}_2 + \boldsymbol{h}_1\boldsymbol{S}_1 + \boldsymbol{h}_2\boldsymbol{S}_2. \tag{2.9}$$

The integration over the '2' sublattice is performed by completing the square, giving the following equations for the scaling of the effective interactions and fields of the remaining variables:

$$\mathbf{H}' = \mathbf{H}_{11} - \mathbf{H}_{12}\mathbf{H}_{22}^{-1}\mathbf{H}_{21} \tag{2.10}$$

$$\boldsymbol{h}' = \boldsymbol{h}_1 - \mathbf{H}_{12}\mathbf{H}_{22}^{-1}\boldsymbol{h}_2. \tag{2.11}$$

In general, it is possible to add an arbitrary constant to the action (2.6), which does not couple to the spins. This term, C, will scale so as to conserve the absolute value of the free energy, $F\{h\}$, after each decimation. Its scaling under renormalisation is given by

$$C' = C - \boldsymbol{h}_2 \mathbf{H}_{22}^{-1} \boldsymbol{h}_2 + K. \tag{2.12}$$

The constant, K, is independent of the fields, $\boldsymbol{h}$, and is proportional to the logarithm of the determinant which arises from the integration over the '2' sublattice.

The results (2.10) and (2.11) are entirely equivalent to those obtained from scaling the equations of motion, but there is no analogy in the equation of motion method for (2.12); the same equations are implied for any value of the constant, C.

Let us now apply these results to the specific cases mentioned above: the pure ferromagnetic chain, the Sierpinski gasket and the sidechain model.

3. The one-dimensional ferromagnet

To illustrate the method with a simple example we consider a Heisenberg ferromagnet on a chain of L sites in the limit $kL \to \infty$ where the boundary conditions can be neglected. The initial action, taking unit exchange coupling J, is

$$A = \tfrac{1}{2} \sum_{\langle ij \rangle} (S_i - S_j)^2 - \omega \sum_i S_i^2 + 2 \sum_i h_i S_i. \tag{3.1}$$

After decimating the action (3.1) by integrating over the variables on alternate sites (so that the length dilatation factor is $b = 2$), we find with the help of equations (2.10)-(2.12) that the renormalised action is

$$A = \frac{1}{2} \frac{1}{(2-\omega)} \sum_{\langle ij \rangle} (S_i - S_j)^2 - \frac{(4\omega - \omega^2)}{(2-\omega)} \sum_i S_i^2 + \sum_i \left(h_i + \frac{h_{i-1} + h_{i+1}}{(2-\omega)} \right) S_i. \tag{3.2}$$

If we were considering the equations of motion of the Green functions we would be justified in multiplying throughout by $(2-\omega)$; in this formulation, in terms of a generating function, this is not possible. Instead we make a change of variables in the integration to find a scaled action in which the coefficient of $S_i S_{i+1}$ is unity (as it was in the undecimated system). Thus the final form for the renormalised action after rescaling is

$$A' = \tfrac{1}{2} \sum_{\langle ij \rangle} (S_i - S_j)^2 - (4\omega - \omega^2) \sum_i S_i^2 + \sum_i \left(\frac{(2-\omega) h_i + h_{i-1} + h_{i+1}}{(2-\omega)^{1/2}} \right) S_i. \tag{3.3}$$

The final recursion relations for ω and h_i can be read off from (3.3) and are as follows:

$$\omega' = 4\omega - \omega^2 \qquad h_i' = \left(\frac{(2-\omega) h_i + h_{i-1} + h_{i+1}}{(2-\omega)^{1/2}} \right). \tag{3.4}$$

Let us now turn to the inhomogeneous or constant term, C, in the action and the contribution it makes at each stage to the Green functions. Since we shall eventually be interested in the derivatives of $F\{h\}$ with respect to the fields $\{h\}$, rather than the absolute value of $F\{h\}$, there is no need to keep track of the Jacobian of the transformation nor the determinant arising from the integration over the '2' sublattice; both of

these contribute to C but are independent of $\{h\}$. From (2.8) we see that

$$G(k,\omega)=\frac{1}{L}\sum_{rr'}\exp[ik(r-r')]G_{rr'}(\omega) \tag{3.5}$$

$$=\frac{1}{L}\sum_{rr'}\exp[ik(r-r')]\frac{\partial^2}{\partial h_r\partial h_{r'}}iF\{h\} \tag{3.6}$$

$$=\frac{1}{L}\frac{\partial^2}{\partial h_k\partial h_{-k}}iF\{h\}. \tag{3.7}$$

To calculate this derivative consider the contribution of the constant term to $G(k,\omega)$ from the nth decimation. From (2.12) this is equal to

$$I^{(n)}=-\frac{1}{2L}\frac{\partial^2}{\partial h_k\partial h_{-k}}\boldsymbol{h}_2^{(n-1)}\mathbf{H}_{22}^{-1}\boldsymbol{h}_2^{(n-1)}$$

$$=\frac{\partial^2}{\partial h_k\partial h_{-k}}\frac{-1}{2L}\sum_{rr'}h_r^{(n-1)}(H_{22})_{rr'}^{-1}h_{r'}^{(n-1)} \tag{3.8}$$

where $\{h^{(n)}\}$ signifies the effective fields acting on the sites which remain after the nth decimation has been performed.

Using (3.4) n times, it is possible to write the effective field acting on any site, at a position r, as a linear combination of the fields acting on the original zeroth generation chain. We write the result as

$$h_r^{(n)}=\sum_m \mathbb{R}_{r,m}^{(n)}h_m^{(0)}. \tag{3.9}$$

Decomposing $\{h^{(0)}\}$ into its Fourier components and substituting into (3.8) gives

$$I^{(n)}=\frac{\partial^2}{\partial h_k\partial h_{-k}}\frac{-1}{2L}\boldsymbol{h}_2^{(n-1)}\mathbf{H}_{22}^{-1}\boldsymbol{h}_2^{(n-1)}$$

$$=\frac{\partial^2}{\partial h_k\partial h_{-k}}\frac{-1}{2L}\sum_{\substack{rr'\\mm'\\qq'}}\mathbb{R}_{r,m}^{(n-1)}(H_{22})_{rr'}^{-1}\mathbb{R}_{r',m'}^{(n-1)}\exp[i(qm+q'm')]h_qh_{q'}$$

$$=\frac{-1}{L}\sum_{\substack{rr'\\mm'}}\mathbb{R}_{r,m}^{(n-1)}(H_{22})_{rr'}^{-1}\mathbb{R}_{r',m'}^{(n-1)}\exp[ik(m-m')]. \tag{3.10}$$

For the one-dimensional chain, where the inverse of $\mathbf{H}_{22}$ is diagonal and where there is an inversion symmetry so that $-k$ is equivalent to k, (3.10) simplifies to give

$$I^{(n)}=\frac{-1}{(2-\omega^{(n-1)})}\frac{1}{L}\sum_r\left(\sum_m\mathbb{R}_{r,m}^{(n-1)}\exp[ik(m-r)]\right)^2$$

$$=\frac{-2^{-n}}{(2-\omega^{(n-1)})}\left(\sum_m\mathbb{R}_{r,m}^{(n-1)}\exp[ik(m-r)]\right)^2. \tag{3.11}$$

The Green function, $G(k,\omega)$, is found by summing $I^{(n)}$ over n:

$$G(k,\omega)=\sum_{n=1}^{\infty}I^{(n)}. \tag{3.12}$$

The sum within the parentheses of (3.11) can be calculated recursively from equation (3.4), for it is just the renormalised field amplitude that would be found acting on the site r after $(n-1)$ iterations if the field acting on the original chain were periodic with wavevector k, and were of unit amplitude.

Thus denoting the field amplitude after the nth decimation by $h_k^{(n)}$, we find from (3.4) that

$$h_{2k}^{(n)} = \left(\frac{(2-\omega)+2\cos k}{(2-\omega)^{1/2}} \right) h_k^{(n-1)} \tag{3.13}$$

where on the left we write h_{2k} rather than h_k because it is convenient to measure all lengths in units of the renormalised lattice spacing.

As stated above, the initial condition for the iteration is $h_k = 1$. Assembling the above arguments we find that the $I^{(n)}$ as defined above (and their sum) are generated by iterating the following inhomogeneous functional equation:

$$G(k, \omega) = \frac{(2-\omega+2\cos k)^2}{2(2-\omega)} G(2k, 4\omega-\omega^2) - \frac{1}{2(2-\omega)}. \tag{3.14}$$

The solution to this equation is well known and is easily checked to be

$$G(k, \omega) = \frac{1}{\omega - 2(1-\cos k)}. \tag{3.15}$$

The functional equation (3.14) can also be iterated numerically in the complex plane by giving ω a small imaginary part, which acts as a convergence factor by smoothing out the singularities in (3.14) coming from the pole at $\omega = 2$. The convergence of this map to the exact solution for the infinite chain is discussed later when we consider a finite chain segment which only requires a finite number of iterations of equation (3.14). We note here that an additional map is also needed for the spins at the end of a segment if the boundary conditions are to be treated correctly. In the next section we apply the above procedure to derive a set of exact recursion relations for the response of a Sierpinski gasket which, unlike the trivial system used in the illustration, cannot be treated by simpler methods.

4. The functional method for a Sierpinski gasket

The Sierpinski gasket is a useful idealisation of the backbones of the large and infinite clusters in a dilute two-dimensional lattice at the percolation threshold, and has been considered in conjunction with a large number of models. In this section we show how the exact functional form of $G(\boldsymbol{k}, \omega)$ can be found using a method similar to that described above for the one-dimensional chain. The only complications for the gasket are firstly that the system does not have inversion symmetry, and secondly that there are three inequivalent sets of sites for each of which we need independent scaling relations to find the magnetic field amplitudes.

Consider the cluster of figure 1, embedded in a Sierpinski gasket of N sites. If we consider the spin S_0 then we can find its couplings in the renormalised action by eliminating the variables on the smallest triangles (i.e. the sites labelled 5–10). If, as usual, a unit of energy is chosen so that the lowest level bonds are of unit strength, then, after a single decimation, the diagonal term, the bond strength (e.g. J_{02}) and the

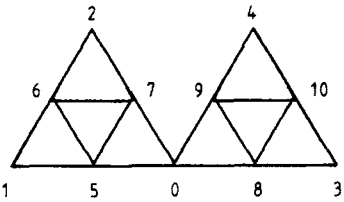


Figure 1. The cluster from a Sierpinski gasket which is used to find the recursion relations of § 4.

renormalised field are as follows:

$$4-\omega'=4-\omega-\frac{4(4-\omega)}{(2-\omega)(5+\omega)} \tag{4.1}$$

$$J'=\frac{6-\omega}{(2-\omega)(5+\omega)} \tag{4.2}$$

$$h_0'=h_0+\frac{(4-\omega)(h_5+h_7+h_8+h_9)+2(h_6+h_{10})}{(2-\omega)(5+\omega)}. \tag{4.3}$$

As in the case of the chain a change of variables in the integration can then be made to find the basic scaling relations for the Sierpinski gasket:

$$\omega'=5\omega-\omega^2 \tag{4.4}$$

$$J'=1 \tag{4.5}$$

$$h_0'=\left(h_0+\frac{(4-\omega)(h_5+h_7+h_8+h_9)+2(h_6+h_{10})}{(2-\omega)(5+\omega)}\right)\left(\frac{(2-\omega)(5+\omega)}{(6-\omega)}\right)^{1/2}. \tag{4.6}$$

The equations (4.4)-(4.6) are now in a form where they can be used iteratively.

Let us consider the contribution of the constant term, C, to $G(\boldsymbol{k}, \omega)$. At the nth iteration we find from (3.8) that

$$I^{(n)}=\frac{-1}{N}\sum_{\substack{rr'\\mm'}}\mathbb{R}^{(n-1)}_{r,m}(H_{22})^{-1}_{rr'}\,\mathbb{R}^{(n-1)}_{r',m'}\exp[\mathrm{i}k(m-m')] \tag{4.7}$$

where N is the number of sites in the original gasket.

Because the matrix $\mathbf{H}_{22}$ is no longer diagonal (though it does remain block diagonal), and because k is not equivalent to $-k$, the simplifications leading to (3.11) are not possible for the gasket. Instead, defining the object

$$\mathbb{R}^{(n)}_{k;r}=\sum_m \exp[\mathrm{i}k(m-r)]\mathbb{R}^{(n)}_{r,m} \tag{4.8}$$

allows us to rewrite (4.7) as

$$I^{(n)}=\frac{-1}{N}\sum_{rr'}\mathbb{R}^{(n-1)}_{k;r}(H_{22})^{-1}_{rr'}\mathbb{R}^{(n-1)}_{-k;r'}\exp[\mathrm{i}k(r-r')]. \tag{4.9}$$

The $\mathbb{R}_k$ will be recognised as the field amplitudes associated with the site r at the nth generation given that the zeroth generation gasket was subjected to a field periodic with wavevector k and of unit amplitude. We are thus led to consider the recursion

relations for the Fourier components of the field. Because the direction of the wavevector $\boldsymbol{k}$ breaks the discrete rotational symmetry of the gasket there are three inequivalent sets of positions for each of which there is an independent field scaling relation (see figure 2). It is also to be noted that there is no simple relationship between $\mathbb{R}_{k;r}$ and $\mathbb{R}_{-k;r}$ except on the original gasket when they are complex conjugates.

Let us denote the field amplitudes acting upon each of the three sublattices by $h_a(\boldsymbol{k})$, $h_b(\boldsymbol{k})$ and $h_c(\boldsymbol{k})$. By this we mean that the effective field acting on the site at position r belonging to the ith sublattice at the nth stage of iteration is

$$h_r = h_i^{(n)}(\boldsymbol{k}) \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}). \tag{4.10}$$

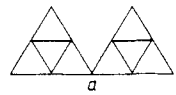
Because distances (and thus wavevectors) will always be measured in terms of the side of the smallest triangle of the gasket, the mapping of k implied by a renormalisation is

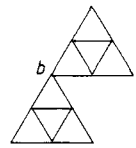
$$\boldsymbol{k}' = 2\boldsymbol{k}. \tag{4.11}$$

If we look at the cluster in figure 1 and assume that the site 0 belongs to the a sublattice, and if the angle that $\boldsymbol{k}$ makes with respect to the line connecting sites 1 and 3 is θ we find that

$$\begin{aligned}
h_0 &= h_a \\
h_5 &= h_a \exp(-\mathrm{i}k\cos\theta) \\
h_8 &= h_a \exp(\mathrm{i}k\cos\theta) \\
h_9 &= h_b \exp[\mathrm{i}k\cos(\theta-\pi/3)] \\
h_6 &= h_b \exp[\mathrm{i}k\sqrt{3}\cos(\theta-5\pi/6)] \\
h_7 &= h_c \exp[\mathrm{i}k\cos(\theta-2\pi/3)] \\
h_{10} &= h_c \exp[\mathrm{i}k\sqrt{3}\cos(\theta-\pi/6)].
\end{aligned} \tag{4.12}$$

On substituting (4.12) into (4.6) we find the renormalised amplitude on sublattice a after a single iteration in terms of the amplitudes of the previous generation.





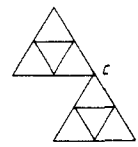


Figure 2. The three inequivalent sublattices which must be considered for the field scaling on a Sierpinski gasket.

The recursion relations for any other sublattice are obtained by rotation by multiples of $2\pi/3$. Thus, for h_b

$$\theta \to \theta + 2\pi/3 \qquad \text{with } a \to b,\ b \to c,\ c \to a \tag{4.13}$$

and for h_c

$$\theta \to \theta + 4\pi/3 \qquad \text{with } a \to c,\ b \to a,\ c \to b. \tag{4.14}$$

We are now in a position to give the recursive method of calculating $G(k, \omega)$ in the limit $N \to \infty$.

(i) Set all the field amplitudes $h_i(\boldsymbol{k})$, $h_i(-\boldsymbol{k})$ equal to unity.

(ii) Decimate to obtain the renormalised parameters using (4.4), (4.6) and (4.12), and at the same time

(iii) calculate the contribution to G by the use of (4.9) which, when explicitly written out, is equal to

$$I^{(n)} = -2\times 3^{-n}\begin{bmatrix} h_a(\boldsymbol{k})\exp(-\mathrm{i}\boldsymbol{k}\boldsymbol{r}_a) \\ h_b(\boldsymbol{k})\exp(-\mathrm{i}\boldsymbol{k}\boldsymbol{r}_b) \\ h_c(\boldsymbol{k})\exp(-\mathrm{i}\boldsymbol{k}\boldsymbol{r}_c) \end{bmatrix}\begin{bmatrix} 3-\omega & 1 & 1 \\ 1 & 3-\omega & 1 \\ 1 & 1 & 3-\omega \end{bmatrix} \times \begin{bmatrix} h_a(-\boldsymbol{k})\exp(\mathrm{i}\boldsymbol{k}\boldsymbol{r}_a) \\ h_b(-\boldsymbol{k})\exp(\mathrm{i}\boldsymbol{k}\boldsymbol{r}_b) \\ h_c(-\boldsymbol{k})\exp(\mathrm{i}\boldsymbol{k}\boldsymbol{r}_c) \end{bmatrix} \frac{1}{(2-\omega)(5-\omega)}. \tag{4.15}$$

(iv) Iterate by returning to step (ii).

(v) Finally we find $G(k, \omega)$ from $I^{(n)}$:

$$G(k, \omega) = \sum_{n=1}^{\infty} I^{(n)}. \tag{4.16}$$

5. Discussion of results for the gasket

Using the method described above, $G(\boldsymbol{k}, \omega)$ has been calculated for various k and θ. The results are plotted in figures 3-5. For small k and ω the results display discrete scaling so that

$$G(\boldsymbol{k}, \omega) = 5^m G(2^m \boldsymbol{k}, 5^m \omega) \tag{5.1}$$

for integral m. This implies that we can write the scaling function as a general periodic function of $\log_2 |\boldsymbol{k}|$, i.e.

$$G(\boldsymbol{k}, \omega) = \sum_m \exp(2\pi \mathrm{i} m \ln|k|/\ln 2)|\boldsymbol{k}|^{-(2-\eta+z)} G_m(\omega/|\boldsymbol{k}|^z, \hat{\boldsymbol{k}}) \tag{5.2}$$

$$= \sum_m |\boldsymbol{k}|^{-(2-\eta+z-2\pi \mathrm{i} m/\ln 2)} G_m(\omega/|\boldsymbol{k}|^z, \hat{\boldsymbol{k}}) \tag{5.3}$$

with $z = \log_2 5$ and $\eta = 2$.

For a random scale-invariant system, such as the infinite cluster at the percolation threshold, the existence of two renormalisation group transformations with an irrational ratio of $\ln b_1$ to $\ln b_2$ implies that only G_0 is non-zero; the system then scales in the conventional manner. For a non-random system, such as the Sierpinski gasket, such a pair of renormalisation group transformations does not exist and the G_m are not subject to any constraint. In figure 3 we show a surface plot of the function

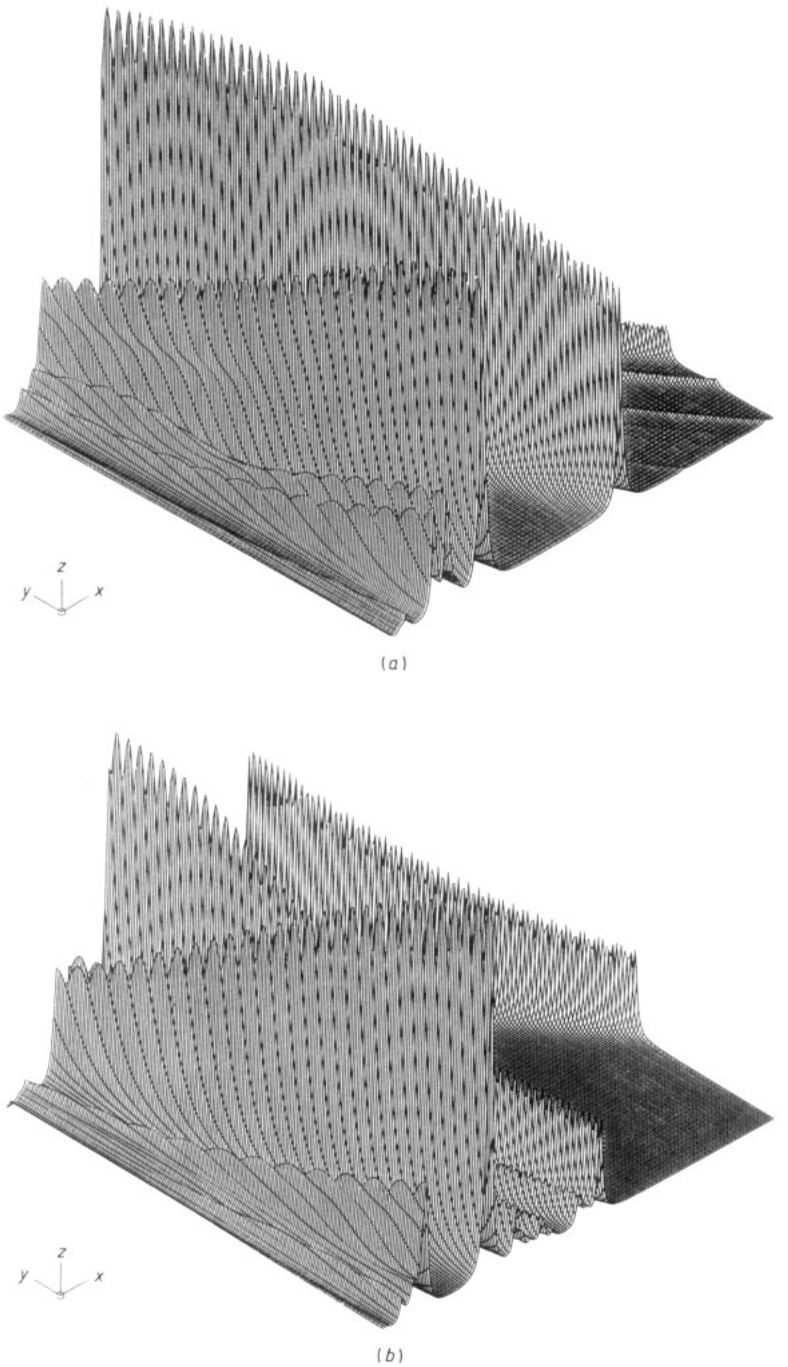
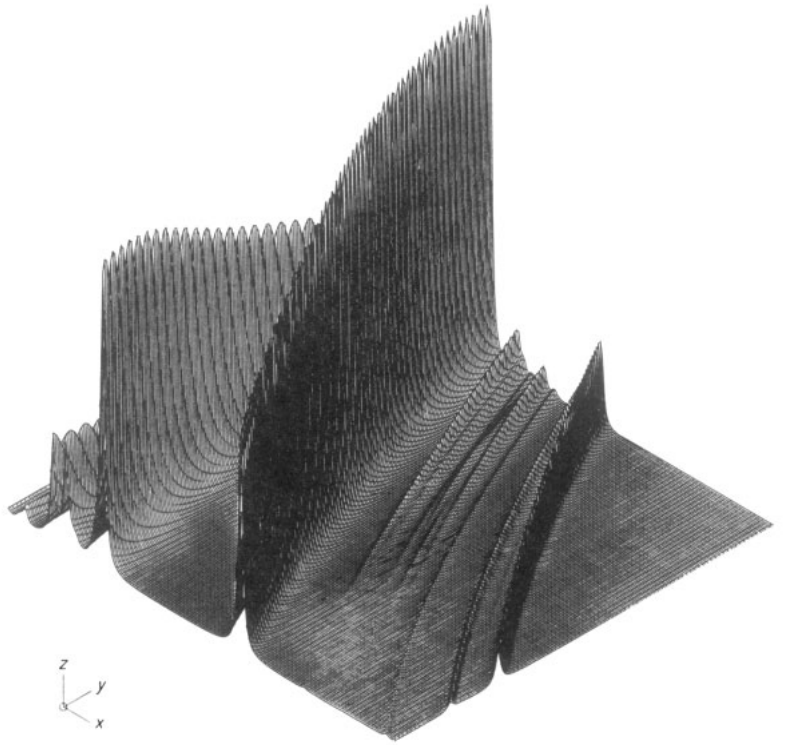
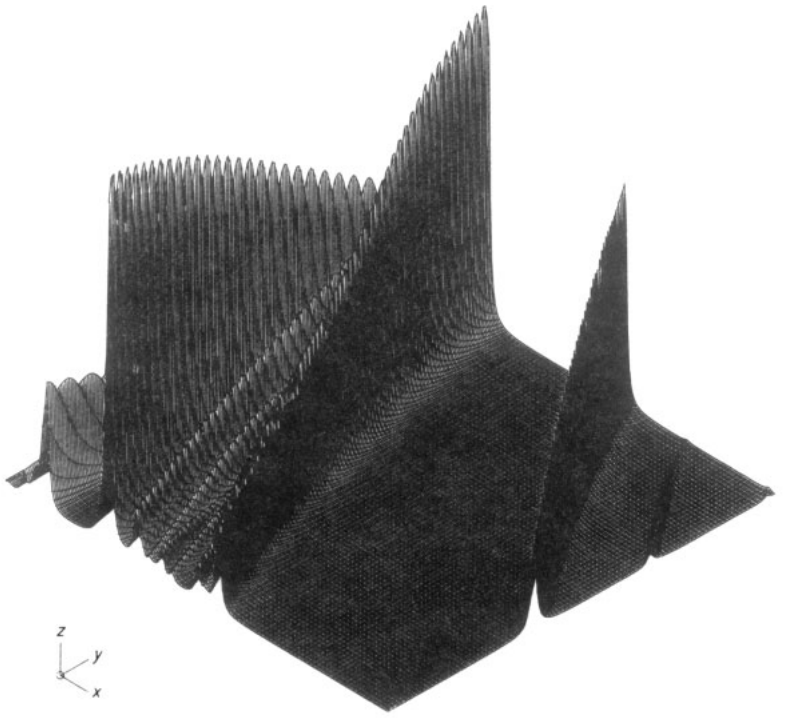


Figure 3. Surface plots of the function $k^{(2-\eta+z)}$ Im $G(\boldsymbol{k}, \omega)$ for a Sierpinski gasket as a function of $\log_2 |\boldsymbol{k}|$ and ω/k^z. The x axis corresponds to ω/k^z for $0 \leq \omega/k^z \leq 2$ and the y axis to $\log_2 |\boldsymbol{k}|$. The result is periodic with period one in $\log_2 |\boldsymbol{k}|$. Each graph covers a half period of the complete discrete scaling function. Thus (*a*) and (*c*) correspond to $0.05 \leq k \leq 0.05\sqrt{2}$ and are two different views of the same surface. (*b*) and (*d*) are for $0.05\sqrt{2} \leq k \leq 0.1$ and are again the same surface from two viewpoints. For all these surfaces $\theta = 0$.



(c)



(d)

Figure 3. (continued)

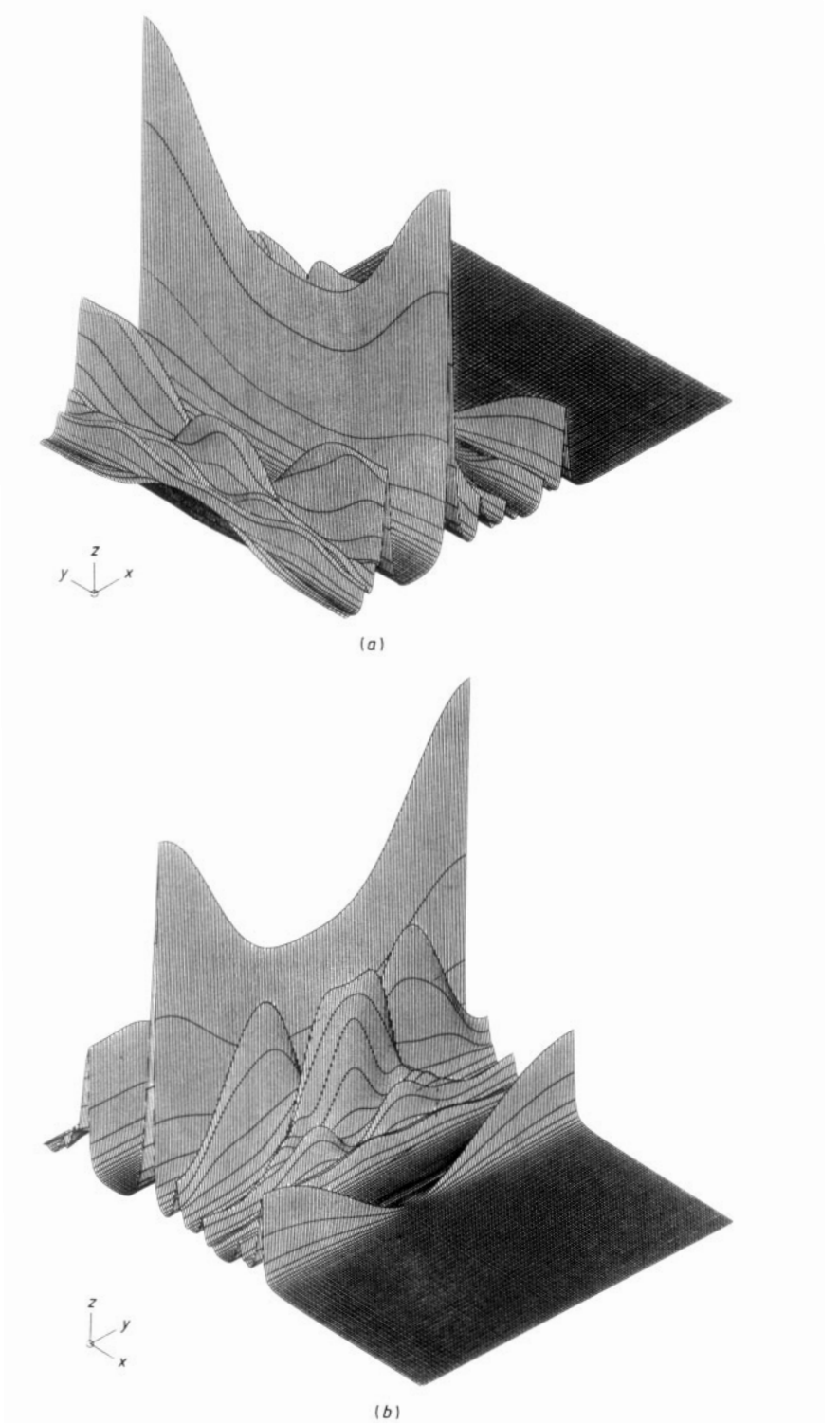


Figure 4. Surface plots of Im $G(\boldsymbol{k}, \omega)$ for a Sierpinski gasket as a function of θ and ω/k^z. The x axis corresponds to ω/k^z for $0 \leq \omega/k^z \leq 2$ and the y axis θ for $0 \leq \theta \leq 60°$. $|\boldsymbol{k}| = 0.07$.

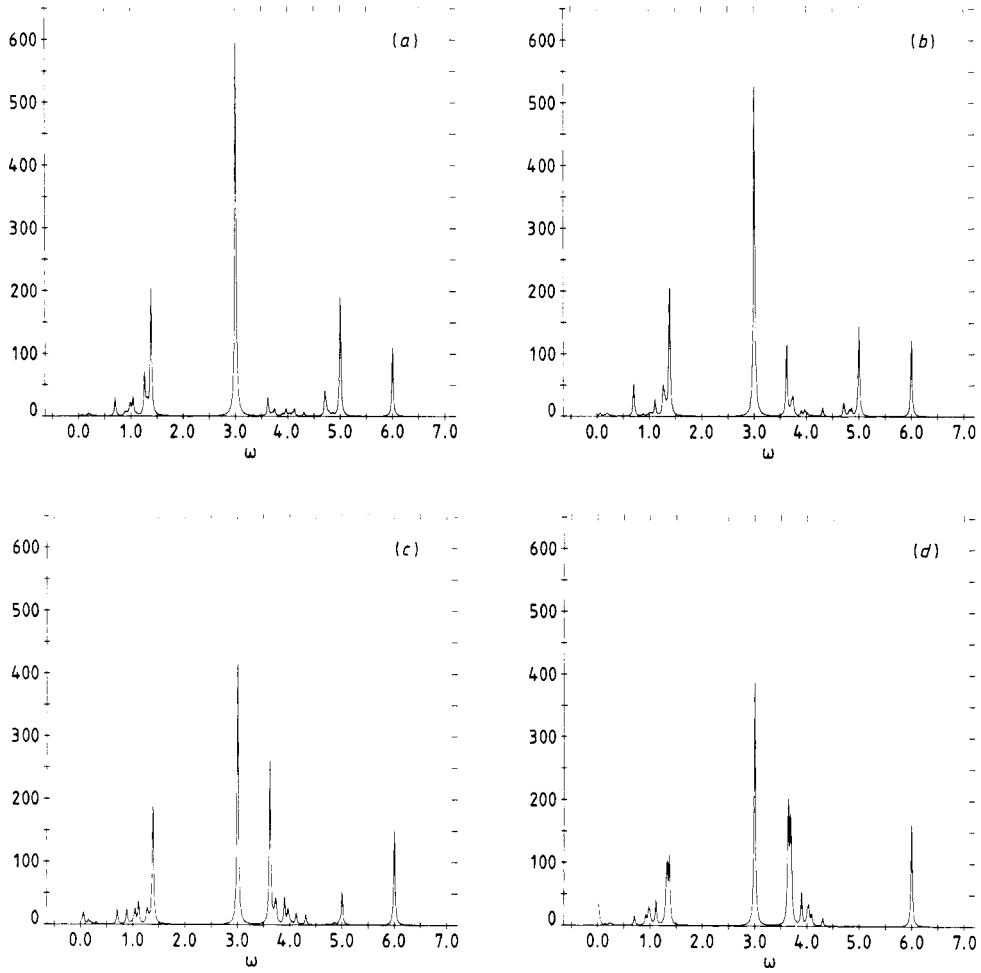


Figure 5. Plots of the dynamic structure factor, $\mathrm{Im}(G(k, \omega))$, of a Sierpinski gasket (as a function of ω), for $|k| = 2.0$ and with θ values of (*a*) 0°, (*b*) 10°, (*c*) 20° and (*d*) 30° respectively. The finite width of the peaks is due to the finite imaginary part given to ω. The vertical scale is arbitrary but consistent between plots.

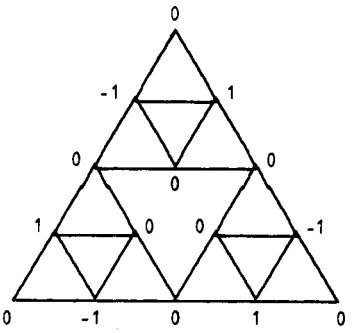


Figure 6. An unnormalised wavefunction for a Sierpinski gasket for $\omega = 5$. There will be no scattering from this eigenstate for $\theta = 30°$ because of the action of a selection rule (§ 5).

$|\boldsymbol{k}|^{(2-\eta+z)}$ Im $G(\boldsymbol{k}, \omega)$ plotted against $\log_2 |\boldsymbol{k}|$ and ω/k^z in the limit of small $|\boldsymbol{k}|$ and ω, for $0 \leqslant \omega/|\boldsymbol{k}|^z \leqslant 2$ and with $\theta = 0$.

All the figures have been calculated by giving ω a finite imaginary part which gives all the peaks their width. In the limit of the imaginary part of ω going to zero the response becomes a sum of delta functions as a consequence of the fact that the support of the spectrum forms a Cantor set, as was shown by Domany *et al* (1983). This is rather different from what is expected on a randomly diluted lattice where local fluctuations produce a much less singular functional form. Thus the result derived above cannot be expected to be a reliable guide to the dynamic structure factor of random clusters.

It is interesting to note that there is a strong angular dependence to the structure factor which has a sixfold symmetry. To show this we have calculated Im $G(\boldsymbol{k}, \omega)$ for constant $|\boldsymbol{k}|$ but for different θ. The results are plotted in figure 4 with $|\boldsymbol{k}| = 0.07$ and for $0 \leqslant \theta < 60°$.

For certain angles selection rules can act to completely eliminate any response from certain modes. An example is provided by figure 5 which was calculated for $|\boldsymbol{k}| = 2.0$ and varying θ. At $\theta = 30°$ (figure 5(*d*)) the response from the modes at $\omega = 5$ vanishes. This can be understood by writing the Green function in the following way:

$$G(k, \omega) = \sum_m \frac{\langle k \,|\, m\rangle\langle m \,|\, k\rangle}{\omega - \varepsilon_m}. \tag{5.4}$$

The dynamic structure factor, which is found from the imaginary part of G, is equal to

$$S(k, \omega) = \sum_m \langle k \,|\, m\rangle\langle m \,|\, k\rangle \pi\delta(\omega - \varepsilon_m). \tag{5.5}$$

The contribution to the intensity from a particular mode, labelled by m, is thus given by $|\langle k \,|\, m\rangle|^2$. In figure 6 we show an unnormalised eigenfunction corresponding to $\omega = 5$. It is easy to see that for $\theta = 30°$ this matrix element must vanish.

Let us now turn to a model for a dilute chain rather than a dilute two-dimensional lattice.

6. A quasi-one-dimensional model of a dilute chain

In this section we model the effects of dilution on a one-dimensional chain by introducing a quasi-one-dimensional system in which there is a length analogous to the percolative correlation length of random systems. Within this length the system behaves like a pure chain but for large separations the propagation of waves is hindered so that the system can mimic the breaking of a pure chain into finite segments. The model chosen is one with a hierarchical construction. Figure 7 shows the construction of the basic lattice with a correlation length of unity. To create a system with a longer correlation length each bond of the unit correlation length system should be replaced by a chain of length ξ. Because of the method of construction the sidechains have a distribution of lengths (so that Bloch's theorem cannot be applied) and act as scatterers. Their separation introduces a natural length for the decay of correlations on the one-dimensional backbone. For simplicity this distance will be taken to be 2^p, where p is integral. The effect of sidechains of uniform length has been considered by Ziman (1979).

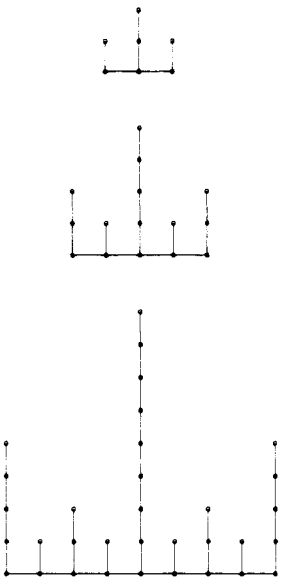


Figure 7. Construction of the quasi-one-dimensional sidechain model of unit correlation length. To form a model of arbitrary correlation length, ξ, each bond should be replaced by a ξ bond segment.

By reversing the process of construction it turns out that the Green functions of the model can be calculated using decimation. Since the main interest of this model is in the Green functions along the backbone, the magnetic fields introduced will only act on these sites. It is equally easy to introduce fields acting on all sites but as this case is of less interest the calculation will not be given here.

The Green functions of the system are calculated in two similar but distinct stages. Firstly the system is decimated until the correlation length (or separation between neighbouring sidechains) is unity. For the second stage it is necessary to solve recursively for the response of the unit correlation length system.

There are four different forms of equations of motion depending on where a spin is situated in the system.

(i) A spin in the interior of a sidechain which has two nearest neighbours:

$$(2-\omega)S_i = \sum_{\text{NN to } i} S_j. \tag{6.1}$$

(ii) A spin at the end of a sidechain with one nearest neighbour:

$$(1-\alpha)S_l = S_{l-1}. \tag{6.2}$$

(iii) A spin in the interior of the backbone when ξ is still greater than one:

$$(2-\omega)S_r = \sum_{\text{NN to } r} S_{r'} + h_r. \tag{6.3}$$

(iv) A spin S_p at the intersection of one of the sidechains and the backbone. These spins have three nearest neighbours, two (S_{p+1}, S_{p-1}) on the backbone and one (S_t) on the sidechain. When the correlation length has been reduced to unity it is not possible to rescale the equations so that all bonds remain of unit strength. Thus we introduce

J_m as the bond strength (starting at an intersection) between spins on the backbone:

$$(3-\beta)S_p = J_m(S_{p-1}+S_{p+1})+S_t+h_p. \tag{6.4}$$

With these equations of motion we now form an action as has been described above.

For a correlation length greater than one we decimate out alternate spins on both the sidechains and the backbone. The sublattice which is eliminated is always chosen so that the sites at the intersection of the sidechains and backbone remain undecimated. The recursion relations for the parameters in the action (after rescaling) are

$$\omega' = 4\omega - \omega^2 \tag{6.5}$$

$$\alpha' = 2\alpha + \omega - \alpha\omega \tag{6.6}$$

$$\beta' = 2\beta + 3\omega - \beta\omega \tag{6.7}$$

$$J'_m = 1 \tag{6.8}$$

$$h'_{2k} = (2-\omega)^{1/2}\left(\frac{2-\omega+2\cos k}{2-\omega}\right)h_k. \tag{6.9}$$

For the first p decimations the contribution to $G(k, \omega)$ comes from sites on the interior of the chain segments. Thus, $I^{(n)}$ for the first p decimations is the same as found for the pure chain above.

When the correlation length has become equal to unity the method of solution must change. We solve for the response of the unit correlation length system by reversing the process of construction in figure 7. The new recursion relations are

$$\omega' = 4\omega - \omega^2 \tag{6.10}$$

$$\alpha' = 2\alpha + \omega - \alpha\omega \tag{6.11}$$

$$\beta' = 3\omega + 2\beta + \frac{2(2-\omega)(1-\alpha)J_m}{(1-\alpha)(3-\beta)-1} - 2 \tag{6.12}$$

$$J'_m = \frac{2(2-\omega)(1-\alpha)J_m^2}{(1-\alpha)(3-\beta)-1} \tag{6.13}$$

$$h'_{2k} = (2-\omega)^{1/2}\left(1+\frac{2\cos kJ_m(1-\alpha)}{(1-\alpha)(3-\beta)-1}\right)h_k. \tag{6.14}$$

The contribution made to the Green function from each decimation is now

$$I^{(n)} = \frac{2^{p-n}h_k^2(1-\alpha)}{(1-\alpha)(3-\beta)-1} \qquad (n>p). \tag{6.15}$$

A computer program has been written to iterate these (exact) equations. The dynamic structure factor has been calculated for various ξ and is plotted in figure 8.

The results for the model can never be regarded as a good approximation to the exact analytic result for the dilute chain (Stinchcombe and Harris 1983). However, they do show a similar crossover between two regimes. For small $k\xi$ the response comes from a broad range of frequencies, but the model gives a result with far more structure than is present in the dilute chain. When $k\xi$ increases there is a spin wave peak because of the long pure chains present, with some additional structure around the peak.

Let us now turn to a solution of the dilute chain using distribution scaling.

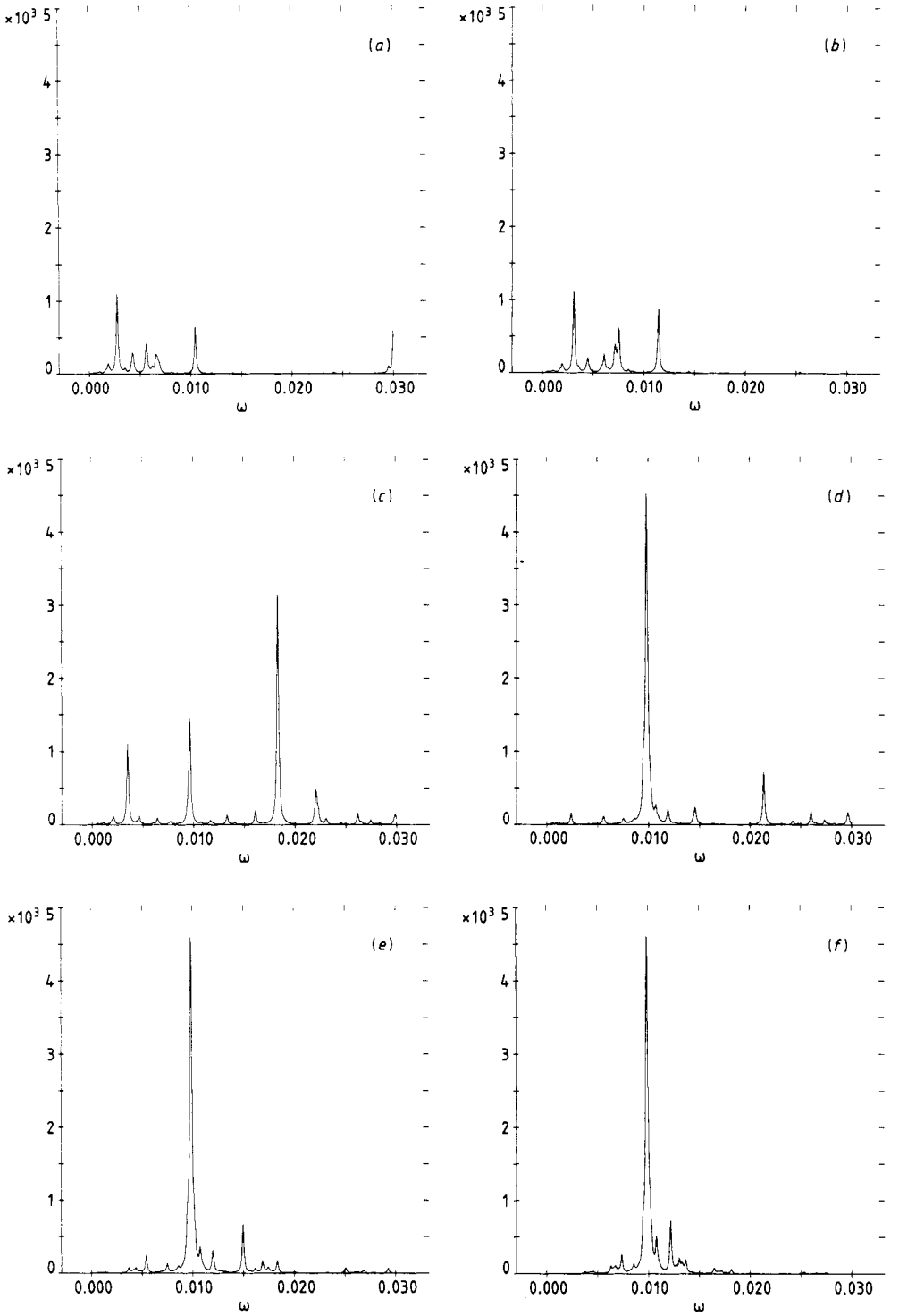


Figure 8. Plots of the response of the backbone of the sidechain model for $k = 0.1$ and for correlation lengths ξ : (*a*) 4, (*b*) 8, (*c*) 16, (*d*) 32, (*e*) 64, (*f*) 128. The response is given in arbitrary units.

7. Application of real space methods to random systems

In the last part of this paper we consider the calculation of the dynamic structure factor of random systems using real space methods. It has proved difficult to make the same kind of crude bond moving and moment averaging approximations to functional recursion relations that have been used to describe the thermodynamics of pure and dilute systems within the framework of the real space renormalisation group (Niemeijer and van Leeuwen 1976, Jayaprakash *et al* 1978 or for a review Stinchcombe 1983b); instead it is necessary to follow the evolution of the full distribution of parameters.

To see the main difficulty in finding approximate functional recursion relations for the response of random systems let us consider figure 9. This shows plots of the dynamic structure factor calculated from a finite number of iterations of (3.14). In the limit of the number of iterations becoming large, the results converge to the exact result for the response of an infinite one-dimensional chain. Unfortunately the convergence is not monotone. For each ω there can be contributions of either sign to the response even though the final result must be positive. The side modes which are visible after the first few iterations are erased by later negative contributions and are thus moved in towards the main spin wave peak; note that the width of the peaks in these plots is due solely to the imaginary part of ω and is not an effect of impurity broadening. These side responses are certainly physical in that they are a consequence of the breakdown of the normal bulk selection rules which only hold in the limit of an infinite system. The presence of these features after a finite number of iterations of the map corresponds to natural finite-size effects. The aim of any recursive method is the iterative solution of successively larger finite-size systems and any such method would therefore be expected to share the property of non-monotone convergence to the infinite system limit. The difficulty in making any approximation for the response is now clear. In general, any such approximation will result in the subtractions which originally cancelled the side modes occurring at frequencies slightly different from the frequency of the original peaks. This is sufficient to destroy the herglotz property of the Green function: the imaginary part of the Green function will oscillate in sign and will be of no use as an approximation for neutron scattering lineshapes. These difficulties do not appear in the usual recursion relations for thermodynamic quantities because in thermodynamic systems one averages over a large number of modes. The analytic properties of quantities such as the free energy are thus tested much less severely (though approximate real space methods have been used successfully in a calculation of the Yang-Lee zeros of the non-random two-dimensional Ising model by Derrida and Flyvbjerg (1985)).

8. The dilute chain

We shall show that in principle it is possible to find an exact scaling method for the response of a dilute one-dimensional ferromagnet using decimation based techniques, rather than the methods of Stinchcombe and Harris (1983) and Maggs and Stinchcombe (1984) from which the exact scaling function is already known. In this paper this new technique will not be used to perform any numerical calculations, but a number of calculations are planned for the near future. The method will involve following the evolution of the distributions of parameters in the part of the action corresponding to

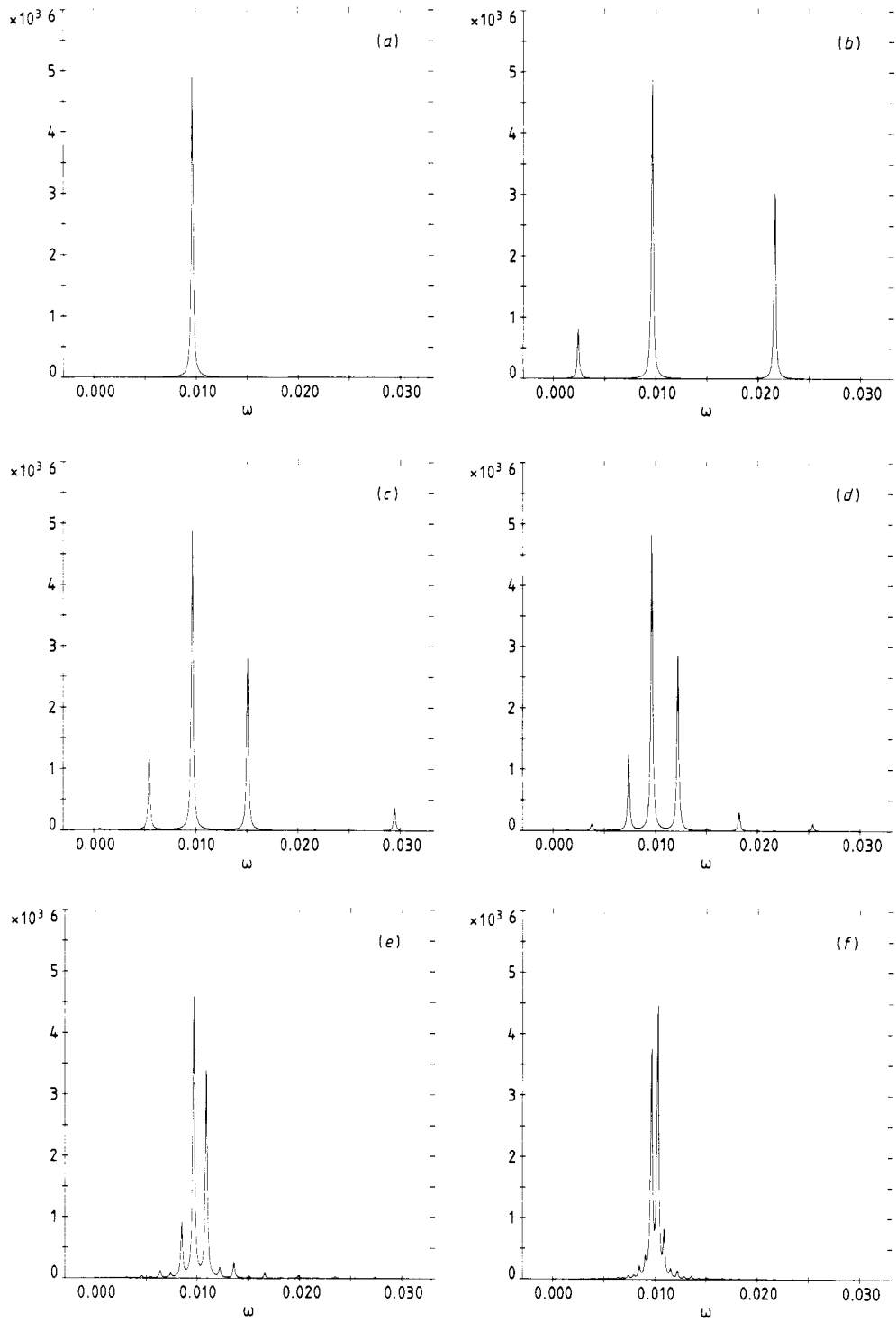


Figure 9. Plots of successive iterates of (3.14) illustrating the non-monotone convergence to the pure limit. For all these graphs $k = 0.1$. The width to the peaks comes from the imaginary part of ω. The number of iterations are (*a*) 5, (*b*) 6, (*c*) 7, (*d*) 8, (*e*) 9, (*f*) 10, (*g*) 11, (*h*) 12.

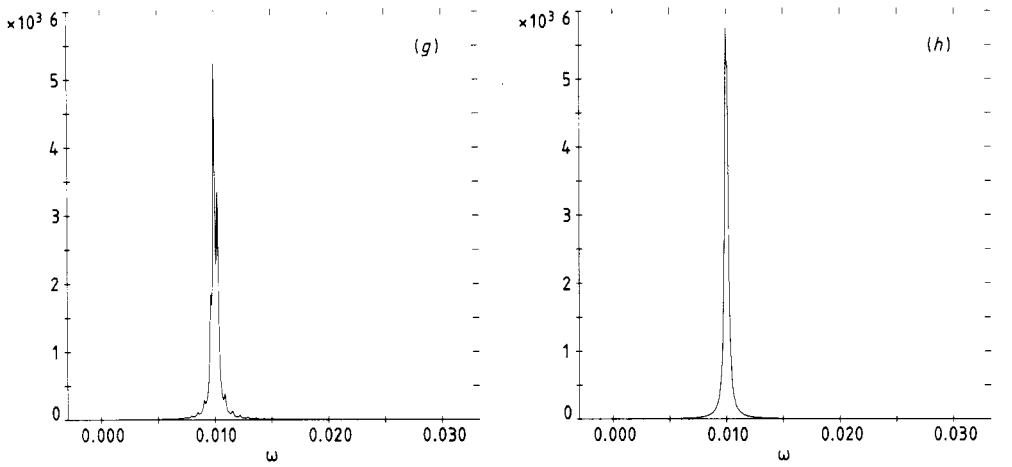


Figure 9. (continued)

the end spins of chain segments. Such distributions are much more difficult to treat than the simple parameter scalings considered until now, so before giving the method for the fully random case it is convenient to consider the calculation of the response of a single chain segment of 2^p bonds. The solution of this problem is similar to the full solution of the dilute chain but is without the complications of the full ensemble of segment lengths.

For such a segment of 2^p bonds it is possible to use decimation to eliminate all but the last two sites by following the evolution of the equations of motion of spins both in the interior and at the ends of the chain. The calculation is terminated by solving for the contribution from the last two sites. This contribution is always important in the scaling limit because the boundary conditions of the chain determine the lowest frequency present on a segment. For the limit $k\xi \to \infty$, boundary conditions are not important, which is why the response of a long chain is well approximated by a large number of iterations of (3.14).

For the interior sites of a 2^p bond segment (with $p > 1$) the equations of motion are

$$(2-\omega)S_i^+ = S_{i+1}^+ + S_{i-1}^+ + h_i^+. \tag{8.1}$$

For the leftmost spin on the segment the corresponding equation is

$$(1-\alpha)S_0^+ = S_1^+ + h_0^+. \tag{8.2}$$

A similar equation holds for the spin on the rightmost site of the segment. The equation for the end sites contains α rather than ω so that the set of equations closes under renormalisation. From these equations we form an action as described above, from which $G(k, \omega)$ can be calculated. Because there are a finite number of sites, only a finite number of stages are needed to find the exact solution for the response.

Integrating over the odd numbered sites (counting the leftmost spin of a segment as zero) and then rescaling the integration variables so that the strength of the renormalised bonds is unity implies the following recursion formulae for ω and α and for h^i, h^l, h^r (which are the field amplitudes for the interior, the leftmost and the rightmost spins respectively on a renormalised segment):

$$\omega' = 4\omega - \omega^2 \tag{8.3}$$

$$\alpha' = 2\alpha + \omega - \alpha\omega \tag{8.4}$$

$$h'^{i}_{2k} = \left(\frac{2-\omega+2\cos k}{(2-\omega)^{1/2}}\right) h^{i}_{k} \tag{8.5}$$

$$h'^{l}_{2k} = \left(\frac{(2-\omega)h^{l}_{k} + e^{ik}h^{i}_{k}}{(2-\omega)^{1/2}}\right) \tag{8.6}$$

$$h'^{r}_{2k} = \left(\frac{(2-\omega)h^{r}_{k} + e^{-ik}h^{i}_{k}}{(2-\omega)^{1/2}}\right). \tag{8.7}$$

For later reference when considering the fully random case let us introduce the following notation for the evolution of the end conditions according to equations (8.4), (8.6) and (8.7):

$$\begin{aligned} h'^{l}_{2k} &= F_1(k, \omega, \alpha, h^{l}_{k}) \\ h'^{r}_{2k} &= F_1(k, \omega, \alpha, h^{r}_{k}) \\ \alpha' &= G_1(\omega, \alpha). \end{aligned} \tag{8.8}$$

Integrating over odd numbered sites of the segment implies that the contribution to the inhomogeneous term from the first $(p-1)$ iterations comes only from the interior sites; $I^{(n)}$ is identical to the $I^{(n)}$ which would be calculated for the infinite chain when $n < p-1$ and can be found by iterating (3.14). After the $(p-1)$th decimation the renormalised system consists of just a pair of coupled degrees of freedom. The final contribution which contains all the information on boundary conditions comes from treating this pair as a special case and integrating over the last two variables remaining in the action. Thus

$$G(k, \omega) = \sum_{n=1}^{p} I^{(n)} \tag{8.9}$$

where $I^{(n)}$ for $n \leq p-1$ is given by the iterates of (3.14) and

$$I^{(p)} = \frac{2(h^{l}_{k})(h^{l}_{-k})(1-\alpha) + (h^{l}_{k})^2 e^{-ik} + (h^{l}_{-k})^2 e^{ik}}{(1-\alpha)^2 - 1} \tag{8.10}$$

and we have used the fact that $h^{l}_{k} = h^{r}_{-k}$.

Let us now turn to the fully random case. Here it is necessary to follow both the deterministic evolution of the couplings of interior sites using (8.3) and (8.5) and a probability distribution for the evolution of the end conditions. The end conditions evolve into a distribution because when the decimation is performed, either a site on the very end of a segment, or a site one lattice spacing in from the end, can be on the surviving sublattice. In the first case the renormalised parameters in the new boundary conditions are given by (8.4), (8.6) and (8.7). In the second case the scaling of the parameters is as follows:

$$1 - \alpha' = (2-\omega)^2 - \frac{(2-\omega)}{(1-\alpha)} - 1 \tag{8.11}$$

$$h'^{l}_{2k} = (2-\omega)^{1/2}\left(\frac{(h^{l}_{k})}{(1-\alpha)} + \frac{(h^{i}_{k})\,e^{2ik}}{(2-\omega)} + (h^{l}_{k})\,e^{ik}\right) \tag{8.12}$$

$$h'^{r}_{2k} = (2-\omega)^{1/2}\left(\frac{(h^{r}_{k})}{(1-\alpha)} + \frac{(h^{i}_{k})\,e^{-2ik}}{(2-\omega)} + (h^{r}_{k})\,e^{-ik}\right). \tag{8.13}$$

For scaling according to equations (8.11)-(8.13) the following notation will be used:

$$h'^{l}_{2k} = F_2(k, \omega, \alpha, h^{l}_{k})$$
$$h'^{r}_{2k} = F_2(k, \omega, \alpha, h^{r}_{k}). \tag{8.14}$$
$$\alpha' = G_2(\omega, \alpha).$$

We are now in a position to write down the equations for the evolution of end conditions of an arbitrary segment.

These equations are only valid for segments sufficiently long that the evolution of the right and left ends of the chain is independent. To ensure this is so we shall only apply them to segments of at least three bonds. All shorter segments must be dealt with individually as a special case, as was done for the last pair of sites in the segment of length 2^p. Thus we find the evolution of the joint probability distribution, D, is

$$D(\alpha', h'^{l}_{2k}, h'^{l}_{-2k}) = \int \frac{\mathrm{d}\alpha}{2}\, \mathrm{d}h^{l}_{k}\, \mathrm{d}h^{l}_{-k} D(\alpha, h^{l}_{k}, h^{l}_{-k})$$
$$\times \{\delta[\alpha' - G_1(\omega, \alpha)]\delta[h'^{l}_{2k} - F_1(k, \omega, \alpha, h^{l}_{k})]\delta[h'^{l}_{-2k} - F_1(-k, \omega, \alpha, h^{l}_{-k})]$$
$$+ \delta[\alpha' - G_2(\omega, \alpha)]\delta[h'^{l}_{2k} - F_2(k, \omega, \alpha, h^{l}_{k})]\delta[h'^{l}_{-2k} - F_2(-k, \omega, \alpha, h^{l}_{-k})]\}. \tag{8.15}$$

We now give the exact recursive method of calculating the Green function for arbitrary p, k and ω.

(1) Set α equal to ω and h_k equal to unity as the initial conditions.

(2) Eliminate all the smallest segments with less than three bonds exactly. The results for the scalings given above assume that the segments are sufficiently long for the left and right end evolutions to be independent.

(3*a*) Decimate to find the contribution made to $G(k, \omega)$. The interior sites of all chains longer than two bonds contribute the following:

$$2^{-n} p^2 [1 - (1-p)^2] \frac{(h^{i}_{k})^2}{(2-\omega)}. \tag{8.16}$$

(3*b*) The number of sites on the left of a segment is $p^3(1-p)$ and the number of sites on the right of a segment is also $p^3(1-p)$.

For a segment of more than two bonds the contribution from the left end is

$$\frac{h^{l}_{k} h^{l}_{-k}}{(1-\alpha)} \tag{8.17}$$

so that the total contribution from end sites (both left and right) is

$$2p^3(1-p) \int D(\alpha, h^{l}_{k}, h^{l}_{-k}) \frac{h^{l}_{k} h^{l}_{-k}}{(1-\alpha)}\, \mathrm{d}\alpha\, \mathrm{d}h^{l}_{k}. \tag{8.18}$$

(4) The decimation which has been performed implies new values for the parameters in the action. Those for the interior are given by (8.3) and (8.5) while the new probability distribution of boundary conditions is given by (8.15). The renormalised bond concentration is $p' = p^2$.

(5) Before performing the next decimation it is again necessary to eliminate all one- and two-bond segments. The contribution of all one-bond segments is

$$2^{-(n-1)}p(1-p)^2\int\begin{pmatrix}h_k^l\\ h_k^r\,\mathrm{e}^{-\mathrm{i}k}\end{pmatrix}\begin{pmatrix}1-\alpha_2 & 1\\ 1 & 1-\alpha_1\end{pmatrix}\begin{pmatrix}h_{-k}^l\\ h_{-k}^r\,\mathrm{e}^{\mathrm{i}k}\end{pmatrix}$$
$$\times\frac{D(\alpha_1,h_k^l,h_{-k}^l)D(\alpha_2,h_k^r,h_{-k}^r)}{(1-\alpha_1)(1-\alpha_2)-1}\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}h_k^l\,\mathrm{d}h_{-k}^l\,\mathrm{d}h_k^r\,\mathrm{d}h_{-k}^r. \tag{8.19}$$

The corresponding contribution of all two-bond segments is

$$2^{-(n-1)}p^2(1-p)^2\int\begin{pmatrix}h_k^l\\ h_k^i\\ h_k^r\,\mathrm{e}^{-2\mathrm{i}k}\end{pmatrix}\begin{pmatrix}1-\alpha_1 & -1 & 0\\ -1 & 2-\omega & -1\\ 0 & -1 & 1-\alpha_2\end{pmatrix}^{-1}\begin{pmatrix}h_{-k}^l\\ h_{-k}^i\,\mathrm{e}^{\mathrm{i}k}\\ h_{-k}^r\,\mathrm{e}^{2\mathrm{i}k}\end{pmatrix}$$
$$\times D(\alpha_1,h_k^l,h_{-k}^l)D(\alpha_2,h_k^r,h_{-k}^r)\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}h_k^l\,\mathrm{d}h_{-k}^l\,\mathrm{d}h_k^r\,\mathrm{d}h_{-k}^r. \tag{8.20}$$

(6) Finally, and as usual,

$$G(k,\omega)=\sum_{n=1}^{\infty}I^{(n)}. \tag{8.21}$$

9. Discussion

As yet no attempt has been made to perform a numerical calculation of the response of a dilute chain by real space decimation methods: they are far more difficult to use than the direct method of configurational averaging of Stinchcombe and Harris (1983) and Maggs and Stinchcombe (1984). However, there is the possibility of applying these methods to systems for which configurational averaging is too difficult to perform. An example would be a Sierpinski gasket with random bond strengths. This would be of interest because distributing bond strengths at random would remove many of the pathologies in the spectrum, and might be expected to provide a better model of the dynamics of random clusters than the regular gasket.

For such a randomised fractal there are three ways in which the calculation of the response could be carried out.

(i) The most simple-minded way would be to generate a large system at random and then solve for the response of the system recursively by storing the renormalised parameters associated with each site after each iteration. While this is not a very sophisticated technique there are considerable time advantages in comparison with standard numerical packages. The recursive method gives the response of an N site system in a time which increases as only $N\log N$. The calculation of the inverse of a large matrix usually takes a time proportional to N^3.

(ii) A method which would be more difficult to implement would be to derive the equations for the evolution of the distributions analytically, as was done above for the chain, and then to follow the evolution of the distributions numerically.

(iii) A method intermediate in complexity between these first two methods is to generate a large system at random and then scale the system once numerically. It should then be possible to extract an approximation to the renormalised coupling distributions and to use this distribution to generate another large random system. Iteration of this method would then overcome the finite-size effects inherent in the first method. A similar method was used by Aoki (1980) in a study of the localisation of electrons in a two-dimensional Anderson model.

10. Conclusion

To calculate the functional form of the dynamic structure factor we have generalised a method first applied to the density of states by Tremblay and Southern (1983) by introducing a transverse field into the equations of motion of the spin operators. This method allowed us to calculate the response function of a number of systems including a pure chain, the sidechain model and a Sierpinski gasket. The result for the Sierpinski gasket did not have the simple dynamic scaling form discussed in § 1 because the lattice only has a discrete scale invariance; rather the response (in the limit of small k and ω) turns out to be a periodic function of $\log_2 |\boldsymbol{k}|$ and also has a strong angular dependence.

We have tried to produce systematically improvable approximations to the Green functions on dilute lattices at arbitrary site and bond concentrations. This has been difficult to accomplish for reasons described in detail in § 7. It would be of great interest to find what analytic properties of a recursion formula must be preserved in order to ensure a positive definite dynamic structure factor: an apparently minor change to an exact recursion formula can have drastic results on the response.

By following the distributions which arise in random systems it is possible, in principle, to find the exact response. However, it is not yet clear that this approach is feasible because the evolution of a multi-dimensional distribution is far more difficult to treat numerically than the simple parameter scaling which has been considered so far. The method will probably only be applicable to hierarchical or fractal models in which the bond strengths are randomised; for such models it is possible to guarantee that the interactions always remain short-ranged. For a random Euclidean lattice, even though the bond strengths fall off rapidly with distance (Maggs 1985, Maggs and Stinchcombe 1986), the correlations in the bonds will be impossible to treat with reasonable computer storage.

References

Aharony A 1985 *Scaling Phenomena in Disordered Systems* ed R Pynn and A Skjeltorp (New York: Plenum)

Alexander S and Orbach R 1982 *J. Physique Lett.* **43** L625

Aoki H 1980 *J. Phys. C: Solid State Phys.* **13** 3369

Chaves C M and Koiller B 1983 *J. Magn. Magnet. Mater.* **38** 183

Derrida B and Flyvbjerg H 1985 *J. Phys. A: Math. Gen.* **18** L313

Derrida B, Orbach R and Kin-Wah Yu 1984 *Phys. Rev.* B **29** 6645

Domany E, Alexander S, Bensimon D and Kadanoff L P 1983 *Phys. Rev.* B **28** 3110

Elliott R J and Pepper D E 1973 *Phys. Rev.* B **8** 2374

Entin-Wohlman O, Alexander S, Orbach R and Kin-Wah Yu 1983 *Phys. Rev.* B **29** 4588

Gonçalves da Silva C E T and Koiller B 1981 *Solid State Commun.* **40** 215

Halperin B I and Hohenberg P C 1969 *Phys. Rev.* **177** 942

Harris C K and Stinchcombe R B 1983 *Phys. Rev. Lett.* **50** 1399

Jayaprakash C, Riedel E K and Wortis M 1978 *Phys. Rev.* B **18** 2244

Langlois J M, Tremblay A M S and Southern B W 1983 *Phys. Rev.* B **28** 218

Lewis S J and Stinchcombe R B 1984 *Phys. Rev. Lett.* **52** 1021

Maggs A C 1985 *D Phil Thesis* University of Oxford

Maggs A C and Stinchcombe R B 1984 *J. Phys. A: Math. Gen.* **17** 1555

—— 1986 to be published

Mandelbrot B B 1977 *Fractals: Form, Chance and Dimension* (San Francisco: Freeman)

Niemeijer Th and van Leeuwen J M J 1976 *Phase Transitions and Critical Phenomena* vol 6, ed C Domb and M S Green (New York: Academic) p 425

Orbach R 1985 *Scaling Phenomena in Disordered Systems* ed R Pynn and A Skjeltorp (New York: Plenum)

Rammal R and Toulouse G 1982 *J. Physique Lett.* **44** L13
Robbins M O and Koiller B 1983 *Phys. Rev.* B **27** 7703
Southern B W and Douchant A R 1985 *Scaling Phenomena in Disordered Systems* ed R Pynn and A Skjeltorp (New York: Plenum)
Stinchcombe R B 1983a *Phys. Rev. Lett.* **50** 200
—— 1983b *Phase Transitions and Critical Phenomena* vol 7, ed C Domb and J L Lebowitz (New York: Academic) p 151
—— 1983c *Highlights of Condensed Matter Physics. Proc. Varenna Summer School* ed F Bassani, F Fumi and M P Tosi (Amsterdam: North-Holland)
—— 1985 *Scaling Phenomena in Disordered Systems* ed R Pynn and A Skjeltorp (New York: Plenum)
Stinchcombe R B and Harris C K 1983 *J. Phys. A: Math. Gen.* **16** 4083
Tahir-Kheli R A 1972 *Phys. Rev.* B **6** 2808
Tremblay A M S and Southern B W 1983 *J. Physique* **44** L843
Ziman T A L 1979 *J. Phys. C: Solid State Phys.* **12** 2645